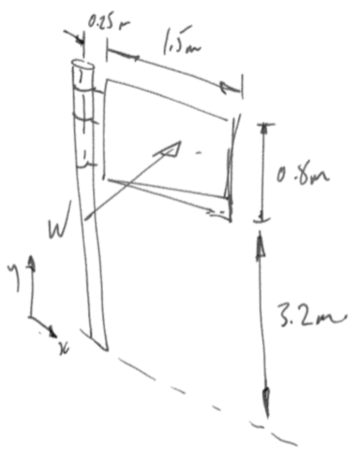
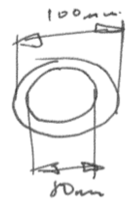
2014-2015 MM2MS2 Exam Solutions

1.

(a)





$p = 2.0kPa$

i) Determine the magnitude of W

$$W = pA = 2\times10^3 \times (1.5\times0.8) = \underline{2.4\times10^3\,N}$$

ii) Calculate the magnitude of the Torque and BM acting about point A

(½ sign, offset to centre of post.)

$$T = Wl = 2.4\times10^3 \times (0.75+0.25) = \underline{2.4\times10^3\,Nm}$$

$$M = Wh = 2.4\times10^3 \times (3.2+0.4) = \underline{8.64\times10^3\,Nm}$$

iii) For a plane stress element at point A, sketch the state of stress and determine the magnitudes of the stresses.

$$\sigma_B = \frac{My}{I} \qquad I = \frac{\pi(D_o^4 - D_i^4)}{64}$$

$$\tau = \frac{Tr}{J} \qquad J = \frac{\pi(D_o^4 - D_i^4)}{32}$$

$$I = \frac{\pi\left((100\times10^{-3})^4 - (80\times10^{-3})^4\right)}{64} = \underline{2.9\times10^{-6}\,m^4} \quad (1)$$

$$J = \frac{\pi\left((100\times10^{-3})^4 - (80\times10^{-3})^4\right)}{32} = \underline{5.8\times10^{-6}\,m^4} \quad (1)$$

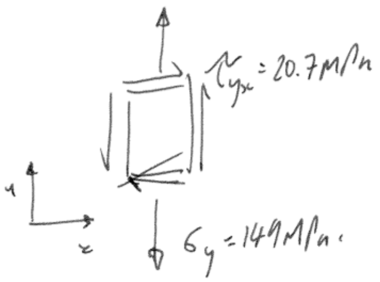
$$\sigma_B = \frac{8.64\times10^3 \times 50\times10^{-3}}{2.9\times10^{-6}} = 1.49\times10^8\,Pa$$

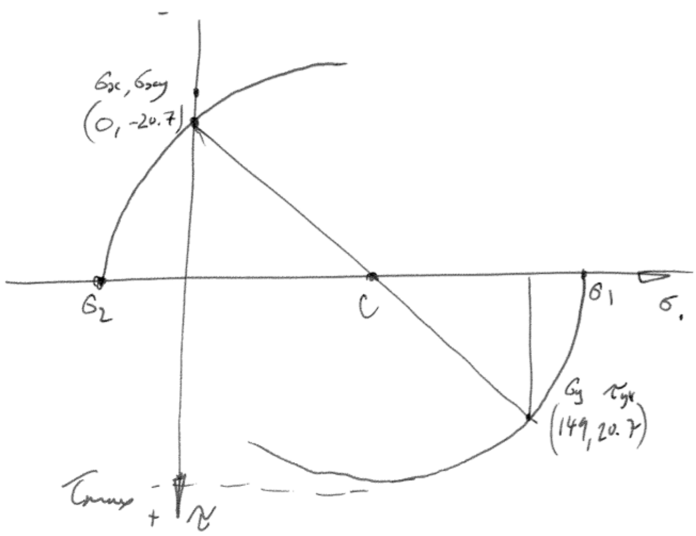
$$= \underline{149\,MPa} \quad (2)$$

$$\tau = \frac{2.4\times10^3 \times 50\times10^{-3}}{5.8\times10^{-6}} = \underline{2.07\times10^7\,Pa}$$

$$= \underline{20.7\,MPa}$$

iii) Determine the magnitude of the principal stresses and maximum shear stress for the stress state on the plane stress element at point A (provide a sketch of Mohr's circle)





$$C = \frac{\sigma_x + \sigma_y}{2} = \frac{149}{2} = \underline{74.5\,MPa}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{-149}{2}\right)^2 - 20.7^2}$$

$$= \sqrt{5550.25 + 428.5}$$

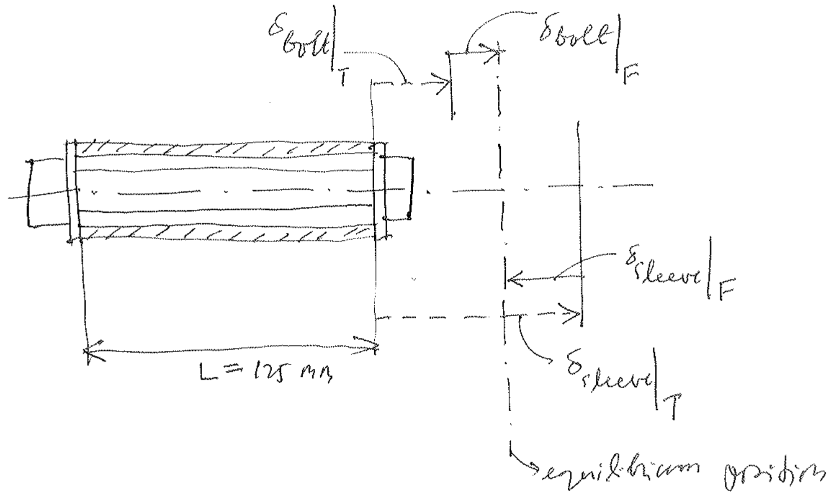
$$= \underline{77.32\,MPa}$$

$$\sigma_1 = C + R = 74.5 + 77.32 = \underline{\underline{151.82\,MPa}}$$

$$\sigma_2 = C - R = 74.5 - 77.32 = \underline{\underline{-2.82\,MPa}}$$

$$\tau = R = \underline{\underline{77.32\,MPa}}$$

2.



Compatibility equation:

$$\delta_{bolt}|_T + \delta_{bolt}|_F = \delta_{sleeve}|_T - \delta_{sleeve}|_F$$

$$\alpha_{bolt} \times \Delta T \times L + \frac{F \times L}{A_{bolt} E_{bolt}} = \alpha_{sleeve} \times \Delta T \times L - \frac{F \times L}{A_{sleeve} E_{sleeve}} \quad \cdots\cdots (i)$$

1

$$F\left(\frac{1}{A_{bolt}E_{bolt}} + \frac{1}{A_{sleeve}E_{sleeve}}\right) = (\alpha_{sleeve} - \alpha_{bolt})\Delta T$$

$$\therefore F = \frac{(23-12)\times 10^{-6} \times (90-25)}{\frac{1}{400\times 10^{-6} \times 200 \times 10^{9}} + \frac{1}{600 \times 10^{-6} \times 73.1 \times 10^{9}}} = 20254.9\ N = 20.3\ kN$$

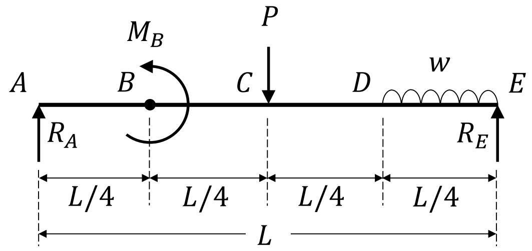
$$\therefore \delta_{tot} = \delta_{bolt}|_T + \delta_{bolt}|_F$$
$$= \left(\alpha_{bolt} \times \Delta T + \frac{F}{A_{bolt}E_{bolt}}\right) \times L$$
$$= \left(12\times 10^{-6} \times 65 + \frac{20.3\times 10^{3}}{80 \times 10^{-6}}\right) \times 125\ mm$$
$$= \underline{0.129\ mm}$$

(b) $\sigma_{bolt} = \frac{20.3\ kN}{400\ mm^2} = \underline{50.75}\ MPa$ (tensile); $\sigma_{sleeve} = \frac{-20.3\ kN}{600\ mm^2} = \underline{-33.83\ MPa}$ (compressive)

(c) consider eq (i)
The results are the same.

3.

(a)



Vertical equilibrium of the beam:

$$R_A + R_E = P + \frac{wL}{4} \qquad (1)$$

[1 mark]

Taking moments about position E:

$$\frac{PL}{2} + \frac{wL^2}{32} = M_B + R_A L$$

$$\therefore R_A = \frac{P}{2} + \frac{wL}{32} - \frac{M_B}{L}$$

Substituting values of P, L, w, and M_B gives:

$$\boldsymbol{R_A = 718.75\ \mathrm{N}}$$

[2 marks]

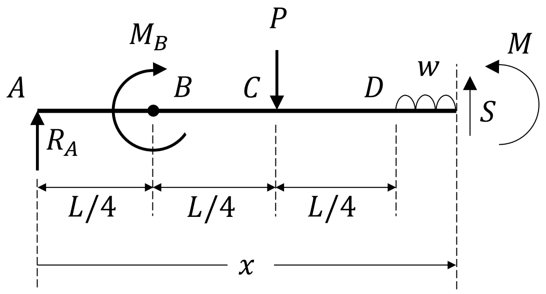
Rearranging (1) for R_E and substituting values for R_A, P, L, w, and M_B gives:

$$\boldsymbol{R_E = 2031.25\ \mathrm{N}}$$

[2 marks]

(b)

Taking the origin at the left-hand end of the beam, sectioning after the last discontinuity and drawing a free body diagram of the left-hand side of the section:



[2 marks]

Taking moments about the section position:

$$M + \frac{w\langle x - \frac{3L}{4}\rangle^2}{2} + P\langle x - \frac{L}{2}\rangle = M_B\langle x - \frac{L}{4}\rangle^0 + R_A x$$

$$\therefore M = M_B\langle x - \frac{L}{4}\rangle^0 + R_A x - \frac{w\langle x - \frac{3L}{4}\rangle^2}{2} - P\langle x - \frac{L}{2}\rangle$$

[2 marks]

Substituting this into the main deflections of beams equation ($EI\frac{d^2y}{dx^2} = M$):

$$EI\frac{d^2y}{dx^2} = M_B\langle x - \frac{L}{4}\rangle^0 + R_A x - \frac{w\langle x - \frac{3L}{4}\rangle^2}{2} - P\langle x - \frac{L}{2}\rangle$$

[1 mark]

Integrating with respect to x:

$$EI\frac{dy}{dx} = M_B\langle x - \frac{L}{4}\rangle + \frac{R_A x^2}{2} - \frac{w\langle x - \frac{3L}{4}\rangle^3}{6} - \frac{P\langle x - \frac{L}{2}\rangle^2}{2} + A \qquad (2)$$

[1 mark]

Integrating with respect to x again:

$$EIy = \frac{M_B\langle x - \frac{L}{4}\rangle^2}{2} + \frac{R_A x^3}{6} - \frac{w\langle x - \frac{3L}{4}\rangle^4}{24} - \frac{P\langle x - \frac{L}{2}\rangle^3}{6} + Ax + B \qquad (3)$$

[1 mark]

Boundary conditions:

(BC1) At $x = 0$, $y = 0$, therefore from (3):

$$B = 0$$

[1 mark]

(BC2) At $x = L$, y = 0, therefore from (3):

$$0 = \frac{9M_B L^2}{32} + \frac{R_A L^3}{6} - \frac{wL^4}{6144} - \frac{PL^3}{48} + AL$$

$$\therefore A = \frac{wL^3}{6144} + \frac{PL^2}{48} - \frac{9M_B L}{32} - \frac{R_A L^2}{6}$$

Substituting values of w, L, P, M_B and R_A into this gives:

$$A = -732.42$$

[1 mark]

From (3), at $x\ = \frac{L}{2}$ (point C):

$$y = \frac{1}{EI}\left(\frac{M_B L^2}{32} + \frac{R_A L^3}{48} + \frac{AL}{2}\right)$$

[1 mark]

Substituting values of E, I, L, M_B, R_A and A into this gives:

$$\boldsymbol{y = -52.95\ \mathrm{mm}}$$

(i.e. downward deflection)

[2 marks]

(c)

From (3), at $x\ = \frac{L}{4}$ (point B):

$$y = \frac{1}{EI}\left(\frac{R_A L^3}{384} + \frac{AL}{4}\right)$$

[2 marks]

Substituting values of E, I, R_A, L and A into this gives:

$$\boldsymbol{y = -35.84\ \mathrm{mm}}$$

(i.e. downward deflection)

[2 marks]

From (2), at $x = \frac{L}{4}$ (point B):

$$\frac{dy}{dx} = \frac{1}{EI}\left(\frac{R_A L^2}{32} + A\right)$$

[2 marks]

Substituting values of E, I, R_A, L and A into this gives:

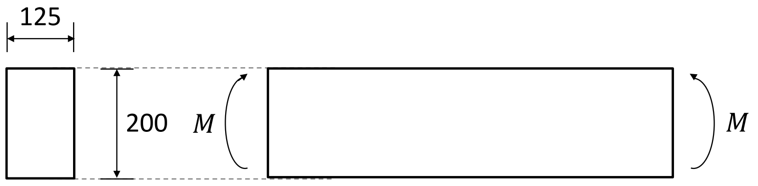
$$\boldsymbol{\frac{dy}{dx} = -0.0656rad = -3.76°}$$

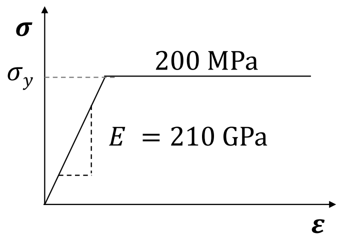
(i.e. small negative gradient)

[2 marks]

4.

(a)





$$I = \frac{bd^3}{12} = \frac{125 \times 200^3}{12} = 83{,}333{,}333.33\ \text{mm}^4$$

[1 mark]

Checking if yielding occurs, first yield will occur when:

$$\sigma = \pm\sigma_y$$

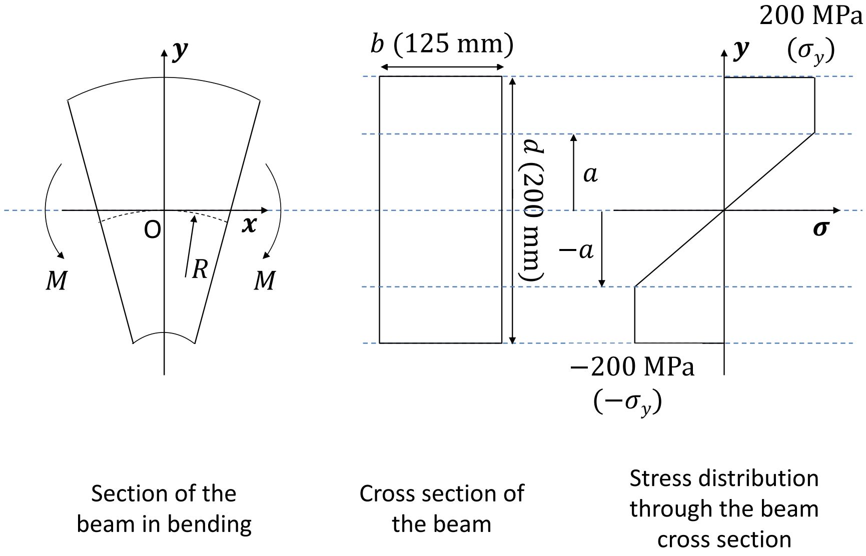
Therefore, at positions $y = \pm 100\ \text{mm}$ $\left(= \frac{200\ \text{mm}}{2}\right)$

$$M_y = \frac{\sigma_y I}{y} = \frac{200 \times 83{,}333{,}333.33}{100} = 166{,}666{,}666.66\ \text{Nm} = 166.66\ \text{kNm}$$

The applied moment (200 kNm) exceeds M_y, therefore yielding will occur.

[2 marks]

Assuming yielding occurs at $y \geq a$ and $y \leq -a$:



Variation of stress with y:

- For a $< y < 100$, $\sigma = 200$ MPa
- For $-$a $< y < a$, $\sigma = \frac{200}{a} y$ MPa
- For $-100 < y < -a$, $\sigma = -200$ MPa

[3 marks]

Moment equilibrium:

(Balance the moments due to stresses in the elastic and plastic regions with the applied moment)

$$M = \int_A y\sigma dA = \int_{-d/2}^{d/2} y\sigma b dy$$

[2 marks]

Due to the symmetry of the stress distribution, this can be rewritten as:

$$M = 2\int_0^{d/2} y\sigma b dy$$

Substituting the elastic and plastic terms for σ into this:

$$M = 2\left\{\int_0^a y\frac{200}{a} y b dy + \int_a^{d/2} y(200) b dy\right\} = 2 \times 200b\left\{\int_0^a \frac{y^2}{a} dy + \int_a^{d/2} y dy\right\}$$

$$= 400b\left\{\left[\frac{y^3}{3a}\right]_0^a + \left[\frac{y^2}{2}\right]_0^{d/2}\right\} = 400b\left\{\left(\frac{a^3}{3a}\right) + \left(\frac{\left(d/2\right)^2}{2} - \frac{a^2}{2}\right)\right\}$$

$$\therefore M = 400b\left\{\frac{d^2}{8} - \frac{a^2}{6}\right\}$$

Substituting the values of b, d and the applied moment, M, into this:

$$200{,}000{,}000 = 400 \times 125\left\{\frac{200^2}{8} - \frac{a^2}{6}\right\}$$

$$\therefore a^2 = 6{,}000$$

$$\therefore a = \pm\sqrt{6{,}000} = \pm\mathbf{77.46\ mm}$$

[2 marks]

Compatibility:

$$\varepsilon = \frac{y}{R} \qquad (1)$$

[1 mark]

At $y = 77.46$ mm, $\sigma = 200$ MPa and since this point is within the elastic range:

$$\varepsilon = \frac{\sigma_y}{E} = \frac{200}{210{,}000} = 9.524 \times 10^{-4}$$

Substituting this into (1) gives:

$$9.524 \times 10^{-4} = \frac{77.46}{R}$$

$$\therefore \boldsymbol{R} = \mathbf{81{,}331.37\ mm = 81.33\ m}$$

[2 marks]

(b)

Unloading is assumed to be entirely elastic. Beam bending equation:

$$\frac{M}{I} = \frac{\sigma}{y}\left(= \frac{E}{R}\right)$$

$$\therefore \frac{\Delta M}{I} = \frac{\Delta \sigma}{y}$$

[2 marks]

Max change in stress ($\Delta\sigma$) will occur at $y = \frac{d}{2} = y_{max}$ ($= \pm 100$ mm).

$$\therefore \Delta\sigma_{max}^{el} = \frac{\Delta M \times y_{max}}{I} = \frac{-M \times y_{max}}{I} = \frac{-200{,}000{,}000 \times \pm 100}{83{,}333{,}333.33}$$

$$= \mp 240 \text{ MPa}$$

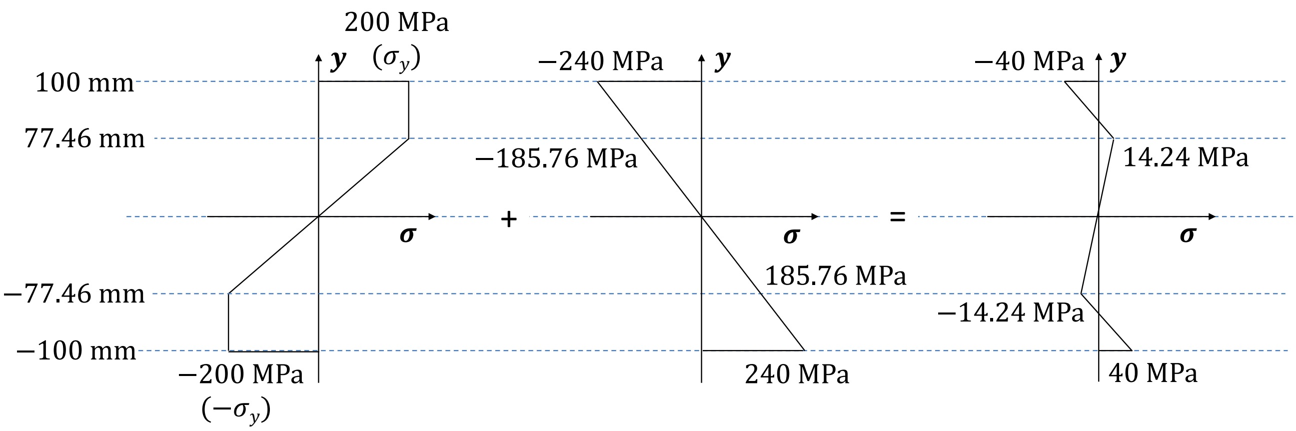
[2 marks]

i.e. at $y = 100$ mm:

$$\Delta\sigma_{max}^{el} = -240 \text{ MPa}$$

and at $y = -100$ mm:

$$\Delta\sigma_{max}^{el} = 240 \text{ MPa}$$



Loading stress distribution

(Elastic) unloading stress distribution

Residual stress distribution

Interpolation of (elastic) unloading line:
At $y = 100$ mm, $\sigma = -240$ MPa
$y = m\sigma + c$
$\therefore 100 = \text{m}\times -240 + 0$
$\therefore \text{m} = -0.417$
At $y = 77.46$ mm, $77.46 = -0.417\times\sigma$
$\therefore \sigma = -185.76$ MPa

[3 marks]

Residual stress is well below yield (200 MPa), so reverse yielding does not occur. At $y = 77.46$ mm, no plastic deformation occurs during loading and unloading,

$$\varepsilon_{residual} = \frac{\sigma_{residual}}{E} = \frac{14.24}{210000} = 6.78 \times 10^{-5}$$

[2 marks]

Also,

$$\varepsilon = \frac{y}{R}$$

[1 mark]

$$\therefore 6.78 \times 10^{-5} = \frac{77.46}{R}$$

$$\therefore \boldsymbol{R = 1,142,477.88\ \mathbf{mm} = 1,142.48\ \mathbf{m}}$$

[2 marks]

5.

(a)

Stress Intensity Factor is given as:

$$K_I = Y\sigma\sqrt{\pi a}$$

Similarly, for K_{II} and K_{III}. Where Y is a function of the crack and component (geometry).

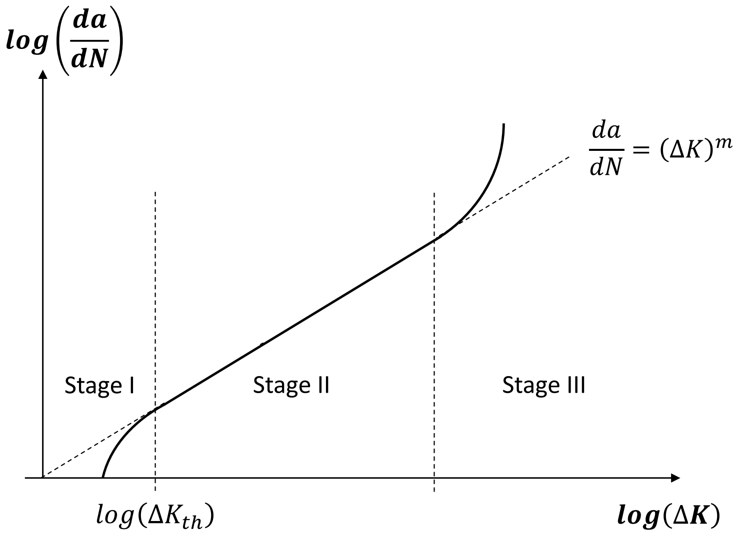
[2 marks]

Paris showed that crack growth can be represented by the following empirical relationship:

$$\frac{da}{dN} = C(\Delta K)^m$$

where C and m are empirically determined material constants. There are 3 stages in the relationship between Crack growth Rate and Stress Intensity Factor, as shown in the following figure.

[2 marks]



<u>Stage I:</u> Below ΔK_{th}, no observable crack growth occurs.

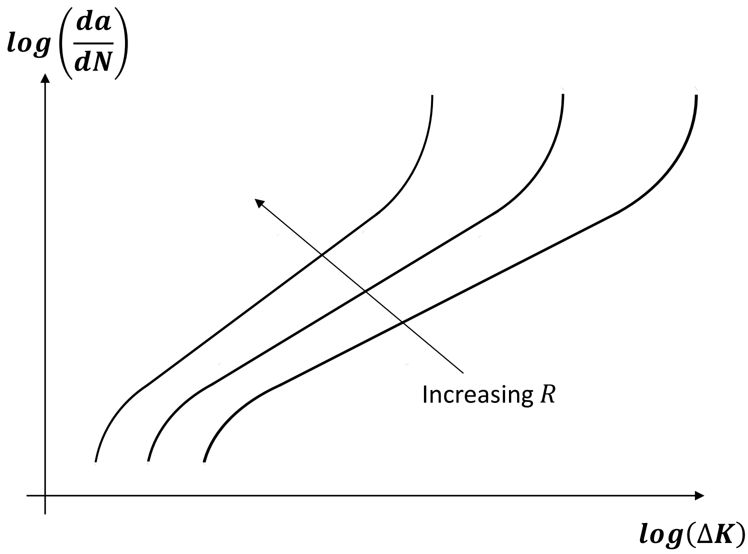
<u>Stage II:</u> This region shows an essentially linear relationship between Crack growth Rate and Stress Intensity Factor (on a log-log scale), where m is the slope and C is the vertical axis intercept.

<u>Stage III:</u> Rapid crack growth occurs, and little life is involved.

[3 marks]

(b)

As mean stress (and therefore R) is increased, fatigue life is decreased as shown in the following figure:



[4 marks]

(c)

$$K_I = \sigma_{nom}\sqrt{\pi a}$$

$$\therefore K_{I_{cr}} = \sigma_{cr}\sqrt{\pi a_{cr}} \qquad (1)$$

[2 marks]

where

$$K_{I_{cr}} = 40MPa\sqrt{m}$$

and

$$\sigma_{cr} = \frac{1}{2}\sigma_y = \frac{1}{2} \times 360 \text{ MPa} = 180 \text{ MPa}$$

[2 marks]

Substituting these values for $K_{I_{cr}}$ and σ_{cr} into (1) gives:

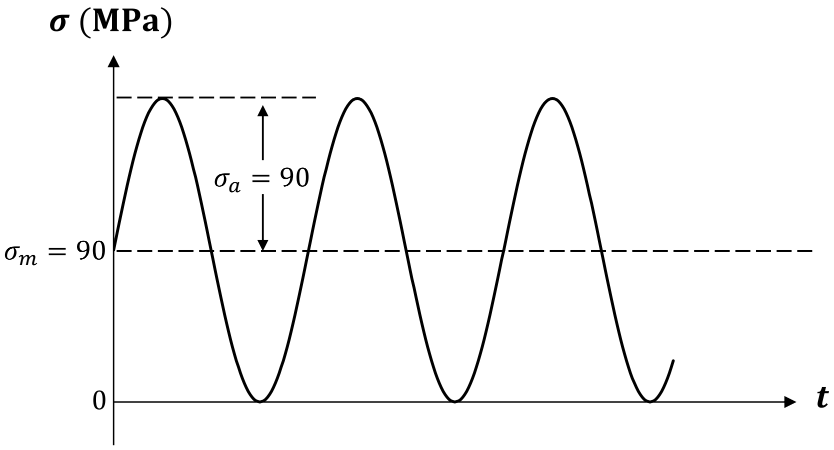
$$40 = 180\sqrt{\pi a_{cr}}$$

$$\therefore \boldsymbol{a_{cr}} = \left(\frac{\mathbf{40}}{\mathbf{180}}\right)^{\mathbf{2}} \times \frac{\mathbf{1}}{\boldsymbol{\pi}} = \mathbf{0.01572\ m} = \mathbf{15.72\ mm}$$

[2 marks]

(d)

The applied load waveform is as shown in the following figure (note that $\sigma_{min} = 0$ MPa, not -30 MPa, as compressive stress does not contribute to crack propagation):



The Paris Law is given as:

$$\frac{da}{dN} = C\Delta K^m \qquad (2)$$

[2 marks]

where:

$$\Delta K = \Delta\sigma\sqrt{\pi a} \qquad (3)$$

[1 mark]

Substituting equation (3) into equation (2) gives:

$$da = C\left(\Delta\sigma\sqrt{\pi a}\right)^m dN$$

$$\therefore dN = \frac{1}{C\left(\Delta\sigma\sqrt{\pi a}\right)^m} da$$

$$\therefore \int_0^N dN = \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m}\int_{a_i}^{a_{cr}} \frac{1}{\left(\sqrt{a}\right)^m} da \left(= \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m}\int_{a_i}^{a_{cr}} \frac{1}{a^{\frac{m}{2}}} da = \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m}\int_{a_i}^{a_{cr}} a^{-\frac{m}{2}} da \right)$$

(i = initial)

$$\therefore [N]_0^N = \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m}\left[\frac{a^{1-\frac{m}{2}}}{1-\frac{m}{2}}\right]_{a_i}^{a_c} \left(= \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^m}\left[\frac{a^{\frac{2-m}{2}}}{\frac{2-m}{2}}\right]_{a_i}^{a_c} \right)$$

$$\therefore N - 0 = \frac{1}{C\left(\Delta\sigma\sqrt{\pi}\right)^{m}}\left(\frac{a_c^{\frac{2-m}{2}} - a_i^{\frac{2-m}{2}}}{\frac{2-m}{2}}\right)$$

$$\therefore N = \frac{2\left(a_c^{\frac{2-m}{2}} - a_i^{\frac{2-m}{2}}\right)}{C\left(\Delta\sigma\sqrt{\pi}\right)^{m}(2-m)} \qquad (4)$$

where:

$$\Delta\sigma = 180 \text{ MPa}$$

$$a_i = 0.001 \text{ m}$$

$$a_c = 0.01572 \text{ m}$$

And typical values for C and m are:

$$m = 2.85$$

and

$$C = 10^{-12} \ \frac{\text{m/cycle}}{\text{MPa}\sqrt{\text{m}}}$$

[2 marks]

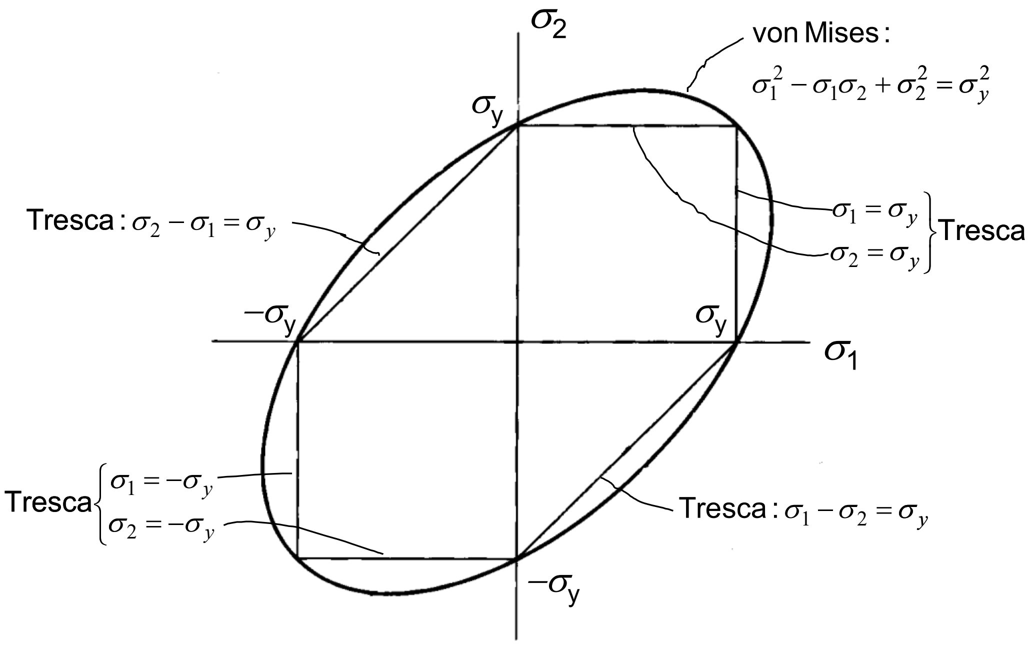
Substituting these values into equation (4) gives:

$$N = \frac{2\left(0.01572^{\frac{2-2.85}{2}} - 0.001^{\frac{2-2.85}{2}}\right)}{10^{-12} \times \left(180 \times \sqrt{\pi}\right)^{2.85} \times (2-2.85)} = \mathbf{2,236,580\ cycles}$$

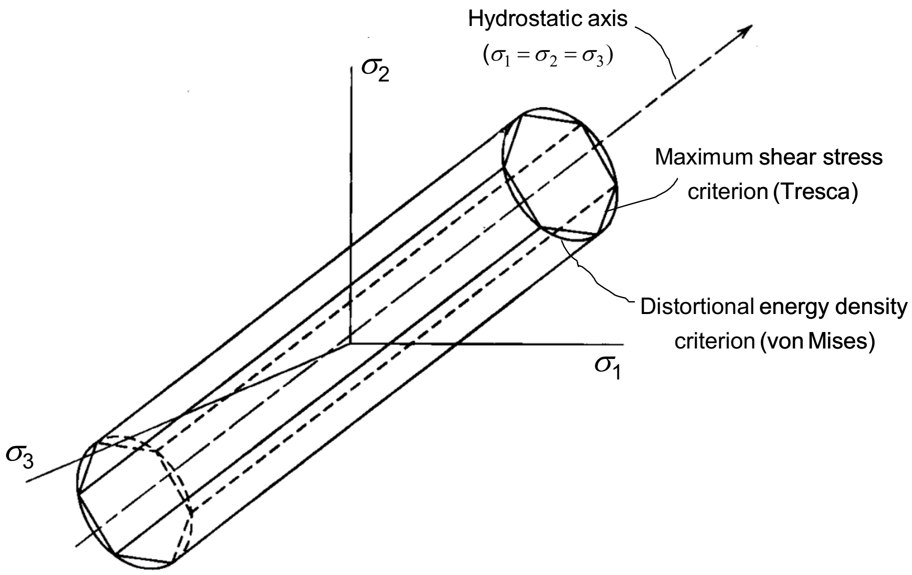
[3 marks]

6.

(a)



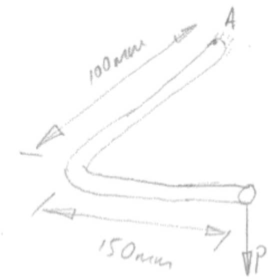
(b)



(c)

See part (b).

(d)



$d = 10mm$
$\sigma_y = 250 MPa$

) By applying the Tresca yield criterion, determine the value of P which will just cause yielding at point A.

$M = 100P$
$T = 150P$

$I = \frac{\pi D^4}{64} = 490.9 mm^4$

$\sigma_x = \frac{My}{I} = \frac{100P \times 5}{490.9} = 1.02P$

$\tau = \frac{Tr}{J} = \frac{150P \times 5}{981.7} = 0.76P$

$J = \frac{\pi D^4}{32} = 981.7 mm^4$

$\sigma_{1,2} = 0.51P \pm \sqrt{(0.51P)^2 + (0.76P)^2}$

$= 0.51P \pm \sqrt{0.26P^2 + 0.58P^2}$

$= 0.51P \pm \sqrt{0.84P^2}$

$= 0.51P \pm 0.92P$

$\Rightarrow \sigma_1 = 1.43P$

$\sigma_2 = -0.41P = \sigma_3$ for plane stress

[6]

Tresca

$\sigma_1 - \sigma_3 = \sigma_y$ at yield

$\Rightarrow \sigma_1 - \sigma_3 = 250$

$1.43P + 0.41P = 250$

$1.84P = 250$

$\Rightarrow P = \underline{135N}$.

137.5

103.1

[5]

) By applying the Von Mises yield criterion, determine the value of P that will just cause yielding at point A.

VM

$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2$ at yield.

$\therefore (1.43P)^2 + (0.41P)^2 + (-0.41P - 1.43P)^2 = 2 \times 250^2$

$2.05P^2 + 0.17P^2 + 3.39P^2 = 125000$

$5.61P^2 = 125000$

$\Rightarrow P = \underline{149N}$.

[5]